PARITY CRITERION AND DEHN TWISTS FOR UNSTABILIZED HEEGAARD SPLITTINGS

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Abstract. We give a parity condition of a Heegaard diagram to show that it is unstabilized. This improves the result of [5]. As an application , we construct unstabilized Heegaard splittings by Dehn twists on any given Heegaard splitting.

1. Introduction

For a closed 3-manifold, a Heegaard splitting is a decomposition of the manifold into two handlebodies. (For a 3-manifold with non-empty boundary, the manifold is decomposed into two compression bodies along their common "plus" boundary.)

The motivation of this paper started from tunnel number one knots. Consider a tunnel number one knot K in S^3 and an unknotting tunnel t for K. Consider two properly embedded arcs γ_1, γ_2 in the exterior of K which have nothing to do with t. Suppose $K \cup \gamma_1 \cup \gamma_2$ gives a genus three Heegaard splitting of exterior of K. Is it irreducible (or unstabilized)? In [4], Kobayashi showed that every genus $g \geq 3$ Heegaard splitting of 2-bridge knot exterior is reducible. So the question is that whether there exists an irreducible genus three Heegaard splitting of a tunnel number one knot exterior which is not 2-bridge. When a non-minimal genus Heegaard splitting is given, in general it is not an easy problem to show that it is irreducible or cannot be destabilized.

However, there are infinitely many examples of manifolds having non-minimal genus irreducible Heegaard splittings. In particular, there exist 3-manifolds having arbitrary high genus strongly irreducible Heegaard splittings ([1], [2]). Casson and Gordon used the **rectangle condition** on Heegaard diagrams to show strong irreducibility of such manifolds. (See ([6], Appendix).) One can also refer to the papers [3], [5], ([8], section 7), ([7], section 7) for the rectangle condition.

Rectangle condition is a condition on Heegaard diagrams for strong irreducibility. One can try to find a condition for irreducibility. Inspired by the example (torus)$\times S^1$, we gave a parity condition in [5], although it is not a weaker condition compared to rectangle condition. It is a condition on two collections of $3g-3$ essential disks giving pants decompositions of the Heegaard surface.

We improve the parity condition of [5]. It is known that a reducible Heegaard splitting of an irreducible manifold is stabilized. Hence, if the manifold under consideration is irreducible, the Heegaard splitting is irreducible. Figure 1. shows the relations of rectangle condition and parity condition for genus $g \geq 2$ Heegaard splittings of irreducible manifolds.

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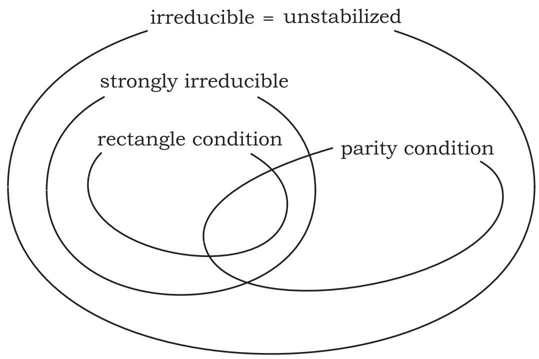

FIGURE 1. Genus $g \geq 2$ Heegaard splittings of irreducible manifolds

Theorem 1.1. *Let $M = H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold M and $\{D_1, D_2 \cdots, D_g\}$ and $\{E_1, E_2, \cdots, E_g\}$ be complete meridian disk systems of H_1 and H_2, respectively.*

If $|D_i \cap E_j| \equiv 0 \pmod 2$ for all the pairs (i, j), then $H_1 \cup_S H_2$ is unstabilized.

As an application, in section 4 we construct unstabilized Heegaard splittings from any given splitting by doing a sequence of Dehn twists.

2. Planar decomposition and pants decomposition of a surface

Let H be a genus $g \geq 2$ handlebody and denote ∂H by S. A collection of essential disks $\{D_1, D_2, \cdots, D_g\}$ in H is called a **complete meridian disk system** for H if the result of cutting H along $\bigcup_{i=1}^{g} D_i$ is a 3-ball. The corresponding result of cutting S by $\bigcup_{i=1}^{g} \partial D_i$ is a planar surface, which is a $2g$-punctured sphere. We call it a **planar decomposition** of S. This terminology was used in ([8], section 7).

In another way, we can decompose H and S into smaller pieces with larger number of essential disks. Suppose a collection of mutually disjoint essential disks $\{D_1, D_2, \cdots, D_{3g-3}\}$ cuts H into 3-balls $B_1, B_2, \cdots, B_{2g-2}$. We can imagine the shape of B_i as a solid pair of pants. Let P_i be the pair of pants $S \cap B_i$ ($i = 1, 2, \cdots, 2g-2$). The decomposition $S = P_1 \cup P_2 \cup \cdots \cup P_{2g-2}$ is called a **pants decomposition** of S.

Let a planar decomposition of S coming from a complete meridian disk system $\{D_1, D_2, \cdots, D_g\}$ be given. We add $2g-3$ more essential disks $\{\bar{D}_{g+1}, \cdots, \bar{D}_{3g-3}\}$ of H to the collection so that $\mathcal{D} = \{D_1, D_2, \cdots, D_g, \bar{D}_{g+1}, \cdots, \bar{D}_{3g-3}\}$ gives rise to a pants decomposition of S. Let $S = P_1 \cup P_2 \cup \cdots \cup P_{2g-2}$ be the new pants decomposition thus obtained. We call $\bar{D}_i$ ($i = g+1, \cdots, 3g-3$) as a **supplementary essential disk** for later use.

Give red color to ∂D_i ($i = 1, 2, \cdots, g$) and blue color to $\partial \bar{D}_j$ ($j = g+1, \cdots, 3g-3$).

Lemma 2.1. *For any P_i, if any two components among the three components of ∂P_i are red and the third is blue, convert the blue-colored component also into red color. Iterate this operation successively until it stops.*

Then all curves constituting the pants decomposition of S become red colors.

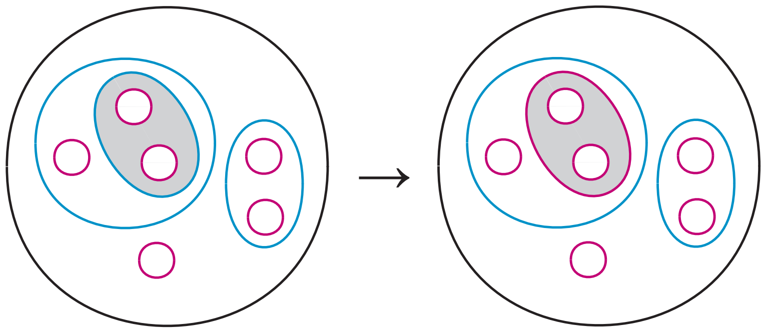

FIGURE 2. The color is changed.

Proof. Originally the planar decomposition of S gave a cutting of S into a $2g$-punctured sphere with red boundaries. We added $2g-3$ essential blue loops on it to get a pants decomposition of S. So there exists a pants having two red loops and one blue loop as its boundary components. Then the color of the blue loop is changed to red. See figure 2. In this way, an innermost blue loop co-bounds a pants with two red loops, and it is changed into red color. Hence, finally all curves come to have red colors. □

Let γ be an essential simple closed curve in S. Note that γ can intersect D_i or $\bar{D}_j$ only at the boundary of the disk ($i = 1, 2, \cdots, g$ and $j = g+1, \cdots, 3g-3$). We assume that γ intersects $(\bigcup_{i=1}^{g} D_i) \cup (\bigcup_{j=g+1}^{3g-3} \bar{D}_j)$ minimally. Let $|\cdot|$ denote the number of elements of a set.

Lemma 2.2. *Suppose* $|\gamma \cap D_i| \equiv 0 \pmod 2$ *for all* ($i = 1, 2, \cdots, g$). *Then* $|\gamma \cap \bar{D}_j| \equiv 0 \pmod 2$ *for all* ($j = g+1, \cdots, 3g-3$).

Proof. Suppose that $\gamma \cap ((\bigcup_{i=1}^{g} D_i) \cup (\bigcup_{j=g+1}^{3g-3} \bar{D}_j)) = \emptyset$. Then γ lives in a pair of pants of the pants decomposition of S. Hence either it is isotopic to ∂D_i for some i or $\partial \bar{D}_j$ for some j. Then it is obvious that $|\gamma \cap \bar{D}_j| \equiv 0 \pmod 2$ for all ($j = g+1, \cdots, 3g-3$).

So we may assume that γ intersects a pair of pants P_k in essential arcs. Let ∂P_k be $l_k^1 \cup l_k^2 \cup l_k^3$. If $|\gamma \cap l_k^1|$ and $|\gamma \cap l_k^2|$ are even numbers, $|\gamma \cap l_k^3|$ should be an even number since $|\gamma \cap (l_k^1 \cup l_k^2 \cup l_k^3)|$ should be an even number. (Every properly embedded arcs in P_k has two endpoints.)

Although the statements of Lemma 2.1 looks irrelevant with this lemma, we use the idea of Lemma 2.1. In the proof of Lemma 2.1, every blue loop, which was the boundary of $\bar{D}_j$, has eventually become a red-colored loop because it co-bounded a pair of pants with two other red loops. Since a red loop has even number of intersection points with γ, we can see that $|\gamma \cap \bar{D}_j| \equiv 0 \pmod 2$ by the conclusion of Lemma 2.1. □

Let D be an essential disk in H. Since a handlebody is an irreducible manifold, we may assume that $D \cap ((\bigcup_{i=1}^{g} D_i) \cup (\bigcup_{j=g+1}^{3g-3} \bar{D}_j))$ is a collection of arcs and the intersection is minimal. The collection of arcs of intersection divides D into subdisks. A subdisk would be a $2n$-gon such as bigon, 4-gon, 6-gon, and so on. Note that bigons are in one-to-one correspondence with outermost disks in D. The

following is a simple observation that is important for the cut-and-connect operation that will be discussed in section 3.

Lemma 2.3. *For all i $(i = 1, 2, \cdots, g)$ and j $(j = g+1, \cdots, 3g-3)$, $|\partial D \cap \partial D_i| \equiv 0 \pmod 2$ and $|\partial D \cap \partial \bar{D}_j| \equiv 0 \pmod 2$.*

Proof. Since any arc of intersection of $D \cap D_i$ has two endpoints, $|\partial D \cap \partial D_i|$ would be an even number. The same holds for $|\partial D \cap \partial \bar{D}_j|$. □

3. Parity condition

Let $H_1 \cup_S H_2$ be a genus $g \geq 2$ Heegaard splitting of a 3-manifold M. Let $\{D_1, D_2, \cdots, D_g\}$ and $\{E_1, E_2, \cdots, E_g\}$ be collections of complete meridian disk systems of H_1 and H_2, respectively. In [5], we gave a parity condition, involving two collections of $3g-3$ essential disks giving pants decompositions of both handlebodies of Heegaard splitting, to be unstabilized. Here we give a more improved condition for an unstabilized Heegaard splitting.

Definition. We say that $H_1 \cup_S H_2$ satisfies the **even parity condition** if $|D_i \cap E_j| \equiv 0 \pmod 2$ for all the pairs $(i, j = 1, 2, \cdots, g)$.

Assume that $H_1 \cup_S H_2$ satisfies the even parity condition. We add $2g-3$ more supplementary essential disks $\{\bar{D}_{g+1}, \cdots, \bar{D}_{3g-3}\}$ of H_1 to the collection $\{D_1, D_2, \cdots, D_g\}$ so that $\mathcal{D} = \{D_1, D_2, \cdots, D_g, \bar{D}_{g+1}, \cdots, \bar{D}_{3g-3}\}$ gives rise to a pants decomposition of S. Also we add $2g-3$ more supplementary essential disks $\{\bar{E}_{g+1}, \cdots, \bar{E}_{3g-3}\}$ of H_2 to the collection $\{E_1, E_2, \cdots, E_g\}$ so that $\mathcal{E} = \{E_1, E_2, \cdots, E_g, \bar{E}_{g+1}, \cdots, \bar{E}_{3g-3}\}$ gives rise to a pants decomposition of S. We assume that all the boundaries of disks meet transversely and minimally.

To simplify the notation, from now on we use the same subscript i for D_i $(1 \leq i \leq g)$ and $\bar{D}_i$ $(g+1 \leq i \leq 3g-3)$. Also we use the same subscript j for E_j $(1 \leq j \leq g)$ and $\bar{E}_j$ $(g+1 \leq j \leq 3g-3)$.

By applying the result of Lemma 2.2, we have the following.

Lemma 3.1. *The parity of number of intersections are as follows.*

1) *For each i, $|D_i \cap \bar{E}_j| \equiv 0 \pmod 2$ for all j.*

2) *For each j, $|\bar{D}_i \cap E_j| \equiv 0 \pmod 2$ for all i.*

3) *For all i and j, $|\bar{D}_i \cap \bar{E}_j| \equiv 0 \pmod 2$.*

Proof. 1) For each i, from the definition of even parity condition, $|D_i \cap E_j| \equiv 0 \pmod 2$ for all j. Then by Lemma 2.2, $|D_i \cap \bar{E}_j| \equiv 0 \pmod 2$ for all j.

2) For each j, from the definition of even parity condition, $|D_i \cap E_j| \equiv 0 \bmod 2$ for all i. Then by Lemma 2.2, $|\bar{D}_i \cap E_j| \equiv 0 \pmod 2$ for all i.

3) For each i, from 2) we can see that $|\bar{D}_i \cap E_j| \equiv 0 \pmod 2$ for all j. Then by Lemma 2.2, $|\bar{D}_i \cap \bar{E}_j| \equiv 0 \pmod 2$ for all j. Then the result 3) follows. (This can be shown by using the result 1) also.) □

Remark 3.2. Lemma 3.1 means that any pair of disks from the collections $\mathcal{D}$ and $\mathcal{E}$ have even number of intersections. So Definition 1 implies ([5], Definition 4).

Now we give the proof of Theorem 1.1.

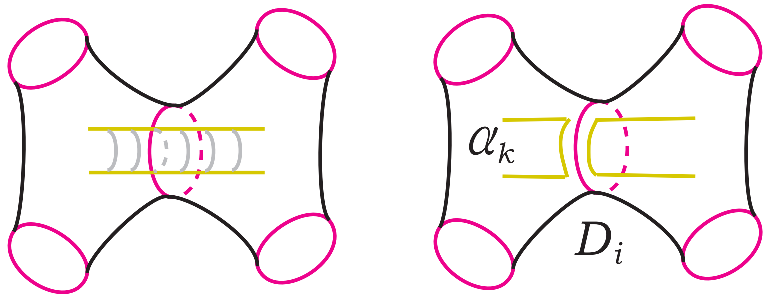

FIGURE 3. α_k lives in a pair of pants.

Proof. (*of Theorem 1.1*)

Suppose that $H_1 \cup_S H_2$ is stabilized. Then there exist essential disks D in H_1 and E in H_2 such that $|D \cap E| = 1$. We may assume that the intersection $D \cap ((\bigcup D_i) \cup (\bigcup \bar{D}_i))$ is a collection of arcs. Cut D by $(\bigcup D_i) \cup (\bigcup \bar{D}_i)$. Then D is divided into subdisks. For any arc, say γ, of intersection $D \cap D_i$ (or $D \cap \bar{D}_i$), two copies of γ, γ_1 and γ_2 are created on both sides of D_i (or $\bar{D}_i$) which are parallel to each other. Connect two endpoints of γ_1 and also connect two endpoints of γ_2 by arcs in S that are parallel and in opposite sides of D_i (or $\bar{D}_i$) to each other as in the Figure 3. We do this cut-and-connect operation for all the arcs $D \cap ((\bigcup D_i) \cup (\bigcup \bar{D}_i))$. Let $\{\alpha_k\}$ be the collection of loops thence obtained from ∂D. Note that each α_k lives in a pair of pants. Some α_k would be isotopic to ∂D_{i_k} and some other α_k be isotopic to $\partial \bar{D}_{i_k}$ and some other α_k would possibly be a trivial loop.

Similarly, from ∂E we obtain a collection of loops $\{\beta_k\}$ by cut-and-connect operations. Some β_k would be isotopic to ∂E_{j_k} and some other β_k would be isotopic to $\partial \bar{E}_{j_k}$ and some other β_k would possibly be a trivial loop.

First we consider the parity of $|D \cap E_j|$ for each j which will be used in the below. Its parity is equivalent to $\sum_k |\alpha_k \cap E_j| \pmod 2$ since in the above cut-and-connect operation two parallel copies γ_1 and γ_2 were created. It is again equivalent to $\sum_k |D_{i_k} \cap E_j| + \sum_k |\bar{D}_{i_k} \cap E_j| + \sum |(\text{trivial loop}) \cap E_j| \pmod 2$. By the even parity condition and Lemma 3.1, it is even. Hence,

$$|D \cap E_j| \equiv 0 \pmod 2$$

By similar arguments, we have the following equalities in $(\bmod 2)$.

$$|D \cap \bar{E}_j| \equiv \sum_k |\alpha_k \cap \bar{E}_j| \equiv \sum_k |D_{i_k} \cap \bar{E}_j| + \sum_k |\bar{D}_{i_k} \cap \bar{E}_j| + \sum |(\text{trivial loop}) \cap \bar{E}_j|$$

By Lemma 3.1, we have

$$|D \cap \bar{E}_j| \equiv 0 \pmod 2$$

Now we have the following equalities in $(\bmod 2)$.

$$|D \cap E| \equiv \sum_k |D \cap \beta_k| \equiv \sum_k |D \cap E_{j_k}| + \sum_k |D \cap \bar{E}_{j_k}| + \sum |D \cap (\text{trivial loop})|$$

By above results, we have

$$|D \cap E| \equiv 0 \pmod 2$$

This is a contradiction since $|D \cap E| = 1$. So we conclude that $H_1 \cup_S H_2$ is unstabilized. □

We give some examples of manifolds admitting a Heegaard splitting satisfying the even parity condition.

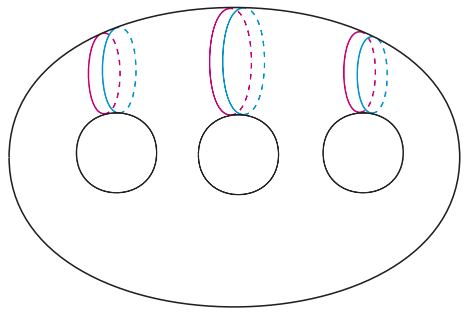

FIGURE 4. Connected sum of three copies of $S^2 \times S^1$, where $D_i \cap E_j = \emptyset$ for all i, j

3.1. **Connected sum of** $S^2 \times S^1$**.** A connected sum of copies of $S^2 \times S^1$ has a Heegaard splitting where each pairs of essential disks in both handlebodies are disjoint (Figure 4). This is a reducible and unstabilized Heegaard splitting.

3.2. (**Torus**) $\times S^1$**.** As an example of irreducible and unstabilized Heegaard splitting satisfying the even parity condition, we consider a genus three Heegaard splitting of (torus)$\times S^1$. Heegaard splittings of manifolds of the form, (surface)$\times S^1$, are classified in [9].

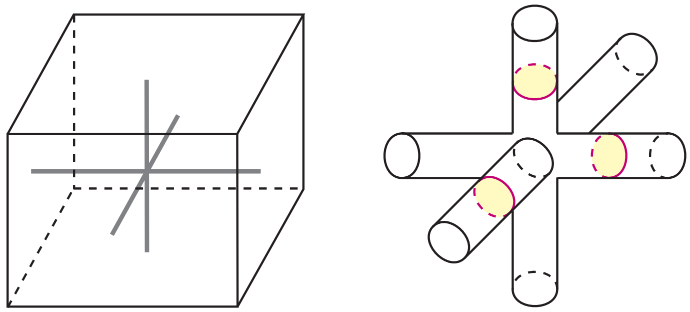

FIGURE 5. Genus three handlebody H_1 in (torus) $\times S^1$

One way of understanding a Heegaard splitting of (torus)$\times S^1$ is as follows. Since (torus)$\times S^1$ is homeomorphic to $S^1 \times S^1 \times S^1$, it can be obtained from a cube by identifying three pairs of opposite faces. Consider the center of the cube and center of each face. Connect the center of the cube with the center of each face by an arc (Figure 5). Take a neighborhood of it and after the identification of opposite sectional disks, we get a genus three handlebody H_1. Figure 5. shows a meridian disk system of H_1.

Now $H_2 = \mathrm{cl}(H_1^c)$ is also a genus three handlebody. Figure 6. shows the boundaries of essential disks of H_2 in ∂H_1. We can see that it satisfies the even parity condition. We can also see that it is weakly reducible. So it is an irreducible and weakly reducible Heegaard splitting.

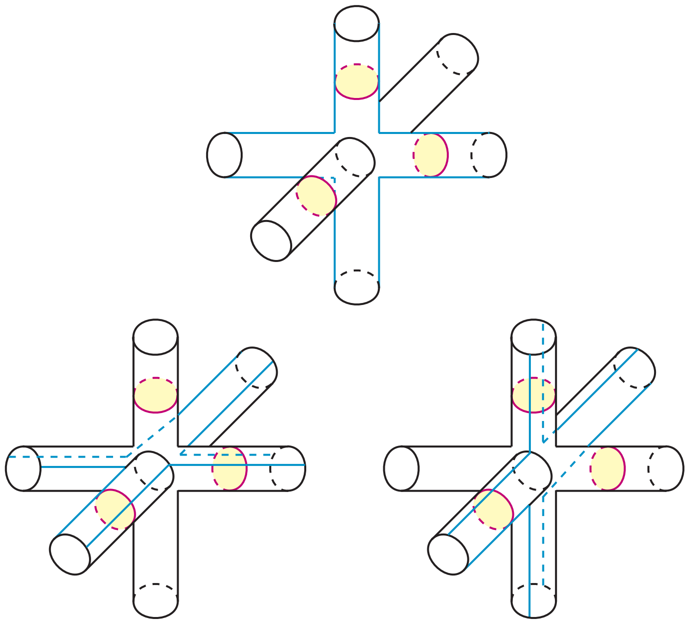

FIGURE 6. Boundaries of essential disks of H_2 in ∂H_1

4. Dehn twist

In this section, we construct unstabilized Heegaard splittings by Dehn twists from a given Heegaard splitting. First we examine the parity of number of intersections of simple closed curves on a surface after a Dehn twist. Let $T_\gamma : S \to S$ be a homeomorphism of a closed surface S, which is a Dehn twist of S along γ.

Lemma 4.1. *For essential simple closed curves $\gamma_1, \gamma_2, \gamma_3$, we have*

$$|T_{\gamma_1}(\gamma_2) \cap \gamma_3| \equiv |\gamma_2 \cap \gamma_3| + |\gamma_1 \cap \gamma_2| \cdot |\gamma_1 \cap \gamma_3| \pmod 2$$

Proof. Before Dehn twist, γ_2 and γ_3 have $|\gamma_2 \cap \gamma_3|$ number of intersection points. After the Dehn twist T_{γ_1}, the number of intersection points is increased by $|\gamma_1 \cap \gamma_2| \cdot |\gamma_1 \cap \gamma_3|$. Since $T_{\gamma_1}(\gamma_2)$ and γ_3 can possibly have inessential intersections (bigons), $|T_{\gamma_1}(\gamma_2) \cap \gamma_3|$ is equivalent to $|\gamma_2 \cap \gamma_3| + |\gamma_1 \cap \gamma_2| \cdot |\gamma_1 \cap \gamma_3|$ by (mod 2). □

By an application of Lemma 4.1, we make unstabilized Heegaard splittings by a single Dehn twist from a given Heegaard splitting satisfying the even parity condition.

Proposition 4.2. *Suppose $H_1 \cup_S H_2$ is an unstabilized Heegaard splitting satisfying the even parity condition with $\mathcal{D} = \{D_1, D_2, \cdots, D_g\}$ and $\mathcal{E} = \{E_1, E_2, \cdots, E_g\}$. Let γ be an essential simple closed curve in S such that $|\gamma \cap E_j| \equiv 0 \pmod 2$ for all j. Alter $\mathcal{D}$ to $\mathcal{D}' = \{D'_1, D'_2, \cdots, D'_g\}$ by a Dehn twist T_γ and leave $\mathcal{E}$ unchanged.*

Then the new Heegaard splitting $H'_1 \cup_{S'} H'_2$ satisfies the even parity condition, hence unstabilized.

Proof. We check the even parity condition for $H'_1 \cup_{S'} H'_2$ by using Lemma 4.1. Note that $\partial D'_i = T_\gamma(\partial D_i)$. By Lemma 4.1,

$$|D'_i \cap E_j| \equiv |D_i \cap E_j| + |\gamma \cap D_i| \cdot |\gamma \cap E_j| \pmod 2$$

In the above equation, $|D_i \cap E_j|$ is even by the even parity condition and $|\gamma \cap E_j|$ is even by the hypothesis of proposition. So $|D_i' \cap E_j|$ is even for all i and j. □

Remark 4.3. Consider a neighborhood $N(\gamma)$ of γ in H_1 such that $N(\gamma) \cap \mathrm{cl}(H_1 - N(\gamma))$ is an annuls whose core is parallel to γ. The Dehn twist T_γ of S is equivalent to removing $N(\gamma)$ from H_1 and attaching a solid torus back so that a meridian of the attaching solid torus is mapped to $\frac{1}{1}$-slope of $\partial N(\gamma)$ (A longitude of the attaching solid torus is mapped to longitude, a parallel of γ.) So in proposition 4.2, the ambient manifold M is changed to a new manifold M' obtained by $\frac{1}{1}$-Dehn filling on γ in M.

Now we are going to get an unstabilized Heegaard splitting satisfying the even parity condition by Dehn twists from a Heegaard splitting which does not satisfy the even parity condition.

Suppose $|D_i \cap E_j|$ is odd for some i and j. For the convenience, assume that $|D_1 \cap E_1|$ is odd. We consider the simple closed curve $\gamma = T_{\partial E_1}(\partial D_1)$ obtained by twisting ∂D_1 along ∂E_1. First we examine the intersection of γ with D_i and E_j.

Lemma 4.4. *The parity of number of intersections of γ with D_i and E_j are as follows* (mod 2)*.*

- $|\gamma \cap D_1| \equiv$ *odd*
- $|\gamma \cap D_i| \equiv$ *odd* $\cdot |D_i \cap E_1| \quad (i = 2, \cdots, g)$
- $|\gamma \cap E_1| \equiv$ *odd*
- $|\gamma \cap E_j| \equiv |D_1 \cap E_j| \quad (j = 2, \cdots, g)$

Proof. • By Lemma 4.1 and assumption, $|\gamma \cap D_1| = |T_{\partial E_1}(\partial D_1) \cap D_1| \equiv |D_1 \cap D_1| + |E_1 \cap D_1| \cdot |E_1 \cap D_1| \equiv \text{odd} \pmod 2$.

• Since D_1 and D_i are disjoint, $|\gamma \cap D_i| = |T_{\partial E_1}(\partial D_1) \cap D_i| \equiv |D_1 \cap D_i| + |E_1 \cap D_1| \cdot |E_1 \cap D_i| \equiv \text{odd} \cdot |D_i \cap E_1| \pmod 2 \ \ (i = 2, \cdots, g)$.

• By assumption, $|\gamma \cap E_1| = |T_{\partial E_1}(\partial D_1) \cap E_1| \equiv |D_1 \cap E_1| + |E_1 \cap D_1| \cdot |E_1 \cap E_1| \equiv \text{odd} \pmod 2$.

• Since E_1 and E_j are disjoint, $|\gamma \cap E_j| = |T_{\partial E_1}(\partial D_1) \cap E_j| \equiv |D_1 \cap E_j| + |E_1 \cap D_1| \cdot |E_1 \cap E_j| \equiv |D_1 \cap E_j| \pmod 2 \ \ (j = 2, \cdots, g)$. □

Now we consider the Dehn twist T_γ of S. Consider the images of ∂D_1 and ∂D_i $(i = 2, \cdots, g)$ after the Dehn twist T_γ. We examine intersections of $T_\gamma(\partial D_1)$ and $T_\gamma(\partial D_i)$ $(i = 2, \cdots, g)$ with E_1 and E_j $(j = 2, \cdots, g)$.

Lemma 4.5. *The parity of number of intersections of $T_\gamma(\partial D_1)$ and $T_\gamma(\partial D_i)$ $(i = 2, \cdots, g)$ with E_1 and E_j $(j = 2, \cdots, g)$ are as follows* (mod 2)*.*

- $|T_\gamma(\partial D_1) \cap E_1| \equiv$ *even*
- $|T_\gamma(\partial D_1) \cap E_j| \equiv$ *even* $\quad (j = 2, \cdots, g)$
- $|T_\gamma(\partial D_i) \cap E_1| \equiv$ *even* $\quad (i = 2, \cdots, g)$

- $|T_\gamma(\partial D_i) \cap E_j| \equiv |D_i \cap E_j| + odd \cdot |D_i \cap E_1| \cdot |D_1 \cap E_j|$ $(i = 2, \cdots, g)$ $(j = 2, \cdots, g)$

Proof. • By Lemma 4.1 and Lemma 4.4 and assumption, $|T_\gamma(\partial D_1) \cap E_1| \equiv |D_1 \cap E_1| + |\gamma \cap D_1| \cdot |\gamma \cap E_1| \equiv \text{odd} + \text{odd} \cdot \text{odd} \equiv \text{even} \pmod 2$.

• By Lemma 4.1 and Lemma 4.4, $|T_\gamma(\partial D_1) \cap E_j| \equiv |D_1 \cap E_j| + |\gamma \cap D_1| \cdot |\gamma \cap E_j| \equiv |D_1 \cap E_j| + \text{odd} \cdot |D_1 \cap E_j| \equiv \text{even} \cdot |D_1 \cap E_j| \equiv \text{even} \pmod 2$ $(j = 2, \cdots, g)$.

• By Lemma 4.1 and Lemma 4.4, $|T_\gamma(\partial D_i) \cap E_1| \equiv |D_i \cap E_1| + |\gamma \cap D_i| \cdot |\gamma \cap E_1| \equiv |D_i \cap E_1| + \text{odd} \cdot |D_i \cap E_1| \cdot \text{odd} \equiv \text{even} \cdot |D_i \cap E_1| \equiv \text{even} \pmod 2$ $(i = 2, \cdots, g)$.

• By Lemma 4.1 and Lemma 4.4, $|T_\gamma(\partial D_i) \cap E_j| \equiv |D_i \cap E_j| + |\gamma \cap D_i| \cdot |\gamma \cap E_j| \equiv |D_i \cap E_j| + \text{odd} \cdot |D_i \cap E_1| \cdot |D_1 \cap E_j| \pmod 2$ $(i = 2, \cdots, g)$ $(j = 2, \cdots, g)$. □

Note that $|T_\gamma(\partial D_i) \cap E_1|$ and $|T_\gamma(\partial D_1) \cap E_j|$ are even for all $i, j = 1, 2, \cdots, g$ and the parity of difference $|T_\gamma(\partial D_i) \cap E_j| - |D_i \cap E_j|$ is equivalent to the parity of $|D_i \cap E_1| \cdot |D_1 \cap E_j|$ for $i, j = 2, \cdots, g$. Let $\mathcal{D}' = \{D'_1, D'_2, \cdots, D'_g\}$ be the collection of essential disks after the Dehn twist T_γ satisfying $\partial D'_i = T_\gamma(\partial D_i)$. Suppose $|D'_i \cap E_j|$ is odd for some i and j. Without loss of generality, we may assume that $|D'_2 \cap E_2|$ is odd. Let $\gamma' = T_{\partial E_2}(\partial D'_2)$. Again we do a Dehn twist $T_{\gamma'}$ of the surface and examine the parities $|T_{\gamma'}(\partial D'_i) \cap E_j|$. By a sequence of Dehn twists in this way, we can get a Heegaard splitting satisfying the even parity condition as the following.

Theorem 4.6. *For any given genus g Heegaard splitting $H_1 \cup_S H_2$ and collections of complete meridian disk systems $\mathcal{D} = \{D_1, D_2, \cdots, D_g\}$ and $\mathcal{E} = \{E_1, E_2, \cdots, E_g\}$, we get an unstabilized Heegaard splitting satisfying the even parity condition after a sequence of at most g Dehn twists.*

More precisely, the sequence of Dehn twists is $T_{\gamma_1}, T_{\gamma_2}, \cdots, T_{\gamma_g}$, where

- $\gamma_1 = T_{\partial E_{j_1}}(\partial D_{i_1})$ *for some i_1 and j_1 with* $|D_{i_1} \cap E_{j_1}| \equiv odd$
- $\gamma_2 = T_{\partial E_{j_2}}(T_{\gamma_1}(\partial D_{i_2}))$ *for some i_2 and j_2 with* $|T_{\gamma_1}(\partial D_{i_2}) \cap E_{j_2}| \equiv odd$
- $\gamma_k = T_{\partial E_{j_k}} \circ T_{\gamma_{k-1}} \circ \cdots \circ T_{\gamma_1}(\partial D_{i_k})$ *for some i_k and j_k with* $|(T_{\gamma_{k-1}} \circ \cdots \circ T_{\gamma_1}(\partial D_{i_k})) \cap E_{j_k}| \equiv odd$ $(k \le g)$

Proof. As before, let $\gamma_1 = T_{\partial E_1}(\partial D_1)$ without loss of generality. Note that $|T_{\gamma_1}(\partial D_i) \cap E_1|$ and $|T_{\gamma_1}(\partial D_1) \cap E_j|$ are even for all $i, j = 1, 2, \cdots, g$ by Lemma 4.5. Let $\mathcal{D}' = \{D'_1, D'_2, \cdots, D'_g\}$ be the collection of essential disks after the Dehn twist T_{γ_1} satisfying $\partial D'_i = T_{\gamma_1}(\partial D_i)$. Let $\gamma_2 = T_{\partial E_2}(\partial D'_2)$ without loss of generality. We examine the parity of $|T_{\gamma_2}(\partial D'_i) \cap E_j|$. Let the notation $\{1, 2, \cdots, \hat{i}, \cdots, g\}$ mean the set $\{1, 2, \cdots, i-1, i+1, \cdots, g\}$. By Lemma 4.5, we have the following.

- $|T_{\gamma_2}(\partial D'_2) \cap E_2| \equiv \text{even}$
- $|T_{\gamma_2}(\partial D'_2) \cap E_j| \equiv \text{even}$ $(j = 1, \hat{2}, \cdots, g)$
- $|T_{\gamma_2}(\partial D'_i) \cap E_2| \equiv \text{even}$ $(i = 1, \hat{2}, \cdots, g)$

- $|T_{\gamma_2}(\partial D_i') \cap E_j| \equiv |D_i' \cap E_j| + \text{odd} \cdot |D_i' \cap E_2| \cdot |D_2' \cap E_j| \quad (i = 1, \hat{2}, \cdots, g)$ $(j = 1, \hat{2}, \cdots, g)$

However, $|T_{\gamma_2}(\partial D_1') \cap E_j| \equiv |D_1' \cap E_j| + \text{odd} \cdot |D_1' \cap E_2| \cdot |D_2' \cap E_j|$ is equal to $|T_{\gamma_1}(\partial D_1) \cap E_j| + \text{odd} \cdot |T_{\gamma_1}(\partial D_1) \cap E_2| \cdot |T_{\gamma_1}(\partial D_2) \cap E_j|$ and it is even because $|T_{\gamma_1}(\partial D_1) \cap E_j|$ and $|T_{\gamma_1}(\partial D_1) \cap E_2|$ are even by Lemma 4.5 again.

Also $|T_{\gamma_2}(\partial D_i') \cap E_1| \equiv |D_i' \cap E_1| + \text{odd} \cdot |D_i' \cap E_2| \cdot |D_2' \cap E_1|$ is equal to $|T_{\gamma_1}(\partial D_i) \cap E_1| + \text{odd} \cdot |T_{\gamma_1}(\partial D_i) \cap E_2| \cdot |T_{\gamma_1}(\partial D_2) \cap E_1|$ and it is even because $|T_{\gamma_1}(\partial D_i) \cap E_1|$ and $|T_{\gamma_1}(\partial D_2) \cap E_1|$ are even by Lemma 4.5.

Hence we can see that $|T_{\gamma_2}(\partial D_1') \cap E_j|$, $|T_{\gamma_2}(\partial D_2') \cap E_j|$, $|T_{\gamma_2}(\partial D_i') \cap E_1|$ and $|T_{\gamma_2}(\partial D_i') \cap E_2|$ are even for all i and j. In this way, as we do sequence of Dehn twists T_{γ_k}, the set of indices of even parity gets bigger and bigger. So finally we get an unstabilized Heegaard splitting satisfying the even parity condition after the sequence of Dehn twists $T_{\gamma_1}, T_{\gamma_2}, \cdots, T_{\gamma_g}$. □

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